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Transient Analysis for Compressible Fluid Flow in Transmission Line by the Method of Characteristics

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Transient analysis for compressible fluid flow has been conducted to evaluate the dynamic characteristics of the dead-ended or volume-terminated transmission lines following a sudden pressure change at its entrance. The two partial-differential equations, based on the conservation of mass, energy and momentum, were derived for the one-dimensional adiabatic compressible flow with friction and entrance head loss. The governing equations describing the present transient-state flow are hyperbolic, and the boundary conditions include a fixed volume termination at the exit and sinusoidal disturbance in the sudden pressure change at tube entrance. The method of characteristics is used to transform the partial differential equation into the particular total differential equations, which can be integrated along the characteristic lines. The present result shows good agreements with the existing results. The effects of tube length, tube diameter and end volume are evaluated on the responses of the pressure and on the damping factor.

Key Words: Dynamic Response, Transmission Line, Method of Characteristics, Characteristic Line, Damping Factor, Lumped Acoustic Element Method

Nomenclature

A	: Cross section area of the pipe
c	: Acoustic velocity
C_c	: Critical damping coefficient
C_k	: Coefficient of entrance loss
D	: Pipe diameter
f	: Friction factor
g	: Acceleration of gravity
H	: Pressure head
k	: Fluid-elastic stiffness coefficient
L	: Pipe length
m	: Effective mass of fluid in the pipe
p	: Pressure
Q	: Flow rate
Re	: Reynolds number
t	: Time
T	: Period of oscillation
u	: Mean flow velocity
V_q	: Fixed volume of the receiver
x	: Axial coordinate

Greek Letters

ζ	: Damping factor
μ	: Viscosity of fluid
ρ	: Density of fluid
ω	: Circular frequency

Superscript

$^+$	: Nondimensional parameter

Subscripts

$_a$	: Based on acoustic velocity
$_e$	: Entrance
$_l$	: Based on lumped acoustic element method
$_{mean}$	: Mean value
$_{res}$	: Sending reservoir
$_{rt}$	: Receiver
$_s$	: Static step input

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1.Introduction

The dynamic responses of transmission lines to a sudden pressure change including sinusoidal inputs have extensively been studied by a numerical method because (a) it has an application to the instrument and control system design (b) it is useful to predict the practical pressure-time relationship and to estimate the performance of the proposed instrument and control system, and (c) it is required to develop a computational method for evaluating the response more rigorously. The information on the dynamic characteristics of dead-ended or volume terminated transmission lines of the system might be useful in a reasonably accurate analysis of the instrument and control system, especially in case of practical pressure measurement at a remote location; *e.g.*, pressure measurement of steam generator in Nuclear Power Plants. An initial time delay and transient attenuation or amplification of pressure at the fixed volume termination will occur following a sudden pressure change.

A theoretical analysis of the frequency response of the pneumatic transmission lines has been performed by Iberall (1950). The analysis was based on incompressible viscous flow and modified to allow compressibility. Some theoretical and/or experimental investigations on the dynamic response of the transmission lines for pneumatic control systems have been conducted (Bradner,1949; Schuder and Binder, 1959; Moise, 1954; Rohmann and Grogen, 1957). The one -dimensional momentum and continuity equations were linearized to obtain an equation of the same form as that for electrical transmission line containing distributed resistance, inductance, and capacitance. Steady sinusoidal disturbances were considered as a boundary condition at the tube entrance (Moise, 1954). Some experimental work on this problem has been performed to validate the analytical method. The response to the step units has been evaluated based on mass, energy and momentum equations with Laplace transformation (Schuder and Binder, 1959). It was found that the application of the Schuder & Binder's theory to the systems involving short, large-diameter lines with small terminal volumes is limited.

In the most previous theoretical studies, the response to the static step inputs or to the sinusoidal disturbances has been analyzed separately. The head losses at the entrance and the exit of transmission lines were not considered to solve the problem and the friction head loss was evaluated based on laminar flow. Practically, it is necessary to evaluate the response to both inputs simultaneously when a pressure change applied at sending end contains both static step inputs and sinusoidal disturbances. In some cases, a resonance will occur due to the coincidence of the frequencies of acoustic waves and sinusoidal inputs. The acoustic waves might be generated as a result of waterhammer (Chaudhry, 1987) and/or a lumped acoustic devices due to the compressibility of a fluid (Kinsler *et al.,* 1982); this lumped acoustic control system can be treated as a Helmholtz resonator. To estimate the response more rigorously, it is required to consider the entrance and exit head losses, especially for relatively short transmission lines and to calculate the friction loss coefficient for laminar or turbulent flow. On this basis, the present study has been handled.

The use of computers for analyzing hydraulic transients has increased considerably in recent years, and the sophisticated numerical methods have been introduced for such analyses (Bulaty and Niessner, 1985; Wiggert, *et al.,* 1985). The numerical methods have permitted the computation of more precise results and have made the analysis of complex systems possible. The governing equations describing the present transient -state flow are hyperbolic partial differential equations. A general solution to the partial differential equations is not available. In the present work, the method of characteristics (Evangelisti, 1969; Abbott, 1966; Holloway and Chaudhry, 1985) is used to transform the partial differential equations into the particular total differential equations. The equations for simulating the transmission line are derived and the boundary conditions for the fixed volume termination including entrance loss effect are developed. The present numerical method is free of restrictions, which can arises from an analytical approach.

The pressure-time relationships at the receiver end with a fixed terminal volume,such as initial time delay and transient attenuation or amplification are illustrated. The effects of tube length, end volume, and tube diameter on the responses have been studied, and frequency response analysis has been conducted by the methods used in linear analysis. The present results have been compared to the results obtained by an existing analytical approach (Schuder and Binder, 1959). The damping factor is evaluated to show the degree of damping on the response of pressure head at the terminal volume.

2. Derivation of the Equations for Transient-State Flow

The piping system as shown in Fig. 1(a) is used for transfer of pressurized fluid and operates under time-varying conditions imposed by a sudden pressure change at its entrance. The receiver represents the internal volume of the instruments or controller and the sending end refers to the connection at the pressure vessel or the output of a pressure transmitter. The fluid in the system is initially static until a sudden pressure input occurs at the sending end. Traditionally, the unsteady behaviour of the fluid flow are analyzed without regard to the motion of the piping for sufficiently rigid pipe. The transients propagate at the acoustic velocity of the fluid in the pipe. In the present analysis, it is assumed that (1) the pressure and temperature changes are small compared to the normal values before the sudden changes, (2) the pipe walls do not stretch regardless of pressure inside the pipe, (3) the pipe line remains full of fluid at all times and the minimum pressure inside the pipe is in excess of the vapor pressure of liquid and (4) acoustic velocity is treated as constants along the pipe and evaluated at the initial temperature and the mean value of the pressure.

The unsteady momentum and continuity equations are applied to a control volume containing a section of transmission line by using the Reynolds transport theorem. In these equations of transient-state flows, there are two dependent variables, pressure p and flow velocity u, and two independent variables, distance x and time t. These governing equations,formed a pair of quasilinear hyperbolic partial differential equations, may be written in a matrix form as

$$\frac{\partial F}{\partial t}+M\frac{\partial F}{\partial x}=E \tag{1}$$

where $$F=\begin{pmatrix} p \\ u \end{pmatrix};\quad M=\begin{bmatrix} u & \rho c^2 \\ \frac{1}{\rho} & u \end{bmatrix};$$

$$E=\begin{pmatrix} 0 \\ -g\sin\theta-\frac{fu|u|}{2D} \end{pmatrix}$$

In the above equations, ρ, j, c and D denote the density of fluid, friction factor, acoustic velocity and the pipe diameter, respectively. The friction factor varies with Reynolds number; *i. e.*, for

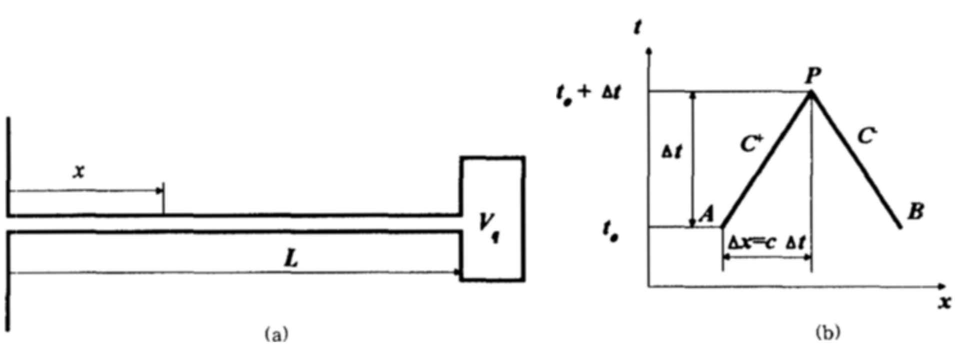


Fig. 1 (a) System nomenclature and (b) characteristic lines in the x-t plane.

laminar flow $f=64/Re$ and for turbulent flow $1/\sqrt{f}=2\ \log_{10}(Re/\sqrt{f})-0.9$ where $Re=\rho u D/\mu$ (Sabersky,*etc*., 1971). Under considerations, the acoustic velocity is obtained by $c=\sqrt{K/\rho}=\sqrt{rRT}$ where K denotes bulk modulus of elasticity of the fluid and R is the gas constant. In most engineering applications, the convective accelerations, $u(\partial p/\partial x)$, $u(\partial u/\partial x)$, are small compared to the other terms. Similarly, the slope term in the right hand side of the momentum equation is usually small and may be neglected.

The method of characteristics has become quite popular and is extensively used for the solution of one-dimensional transient problems.This method has proven to be superior to other numerical methods in several aspects, such as the correct simulation of steep wave fronts, the illustration of wave propagation, the ease of programming and the efficiency of computations. The governing equations are transformed into four ordinary differential equations by the method of characteristics. The interested reader on the method of characteristics is referred to "Applied Hydraulic Transients" by Chaudhry (1987).

Now, let us discuss the physical significance of the characteristics line in the x-t plane as shown in Fig. 1(b). In as much as acoustic velocity, c, is generally constant for a given pipe, dx/dt plots as a straight on x-t plane. These lines on the x-t plane are the "characteristic" lines along which the compatibility equations are valid. Mathematically, these lines divide the x-t plane into two regions, which may be dominated by two different kinds of solution; *i.e*., the solution may be discontinuous across these lines. Physically, they represent the path traversed by a disturbance.

It is a common practice in hydraulic engineering to compute the pressure in the pipeline in terms of head, $H=p/\rho g$, and use the flow rate instead of the flow velocity; *e.g*, $Q=uA(A=\pi D^2/4)$. Eventually, integrating of the compatibility equation (derived from governing equations) along the characteristic lines, $C+(dx/dt=+c)$, leads to

$$(H_P-H_A)+\frac{c}{gA}(Q_P-Q_A)+\frac{f\Delta x}{2gDA^2}Q_A|Q_A|=0 \tag{2}$$

and along C- line, the following negative characteristic equation is obtained;

$$(H_P-H_B)-\frac{c}{gA}(Q_P-Q_B)-\frac{f\Delta x}{2gDA^2}Q_B|Q_B|=0 \tag{3}$$

These two compatibility equations are the basic algebraic relations that describe the transient propagation of pizeometric head and flow in the transmission line.

In order to generalize the present problem, it is convenient to define the following nondimensional parameters:

$$H^+=\frac{H-H_{ref}}{H_s-H_{ref}},\quad Q^+=\frac{Q}{\frac{gA}{c}(H_s-H_{ref})}$$
$$K^+_{res}=\frac{(1+C_k)g}{2c^2}(H_s-H_{ref})$$
$$K^+_{rt}=\frac{(1-C_k)g}{2c^2}(H_s-H_{ref})$$
$$R^+=\frac{f}{2DA}\Delta t|Q|,\quad D^+_a=\frac{cA}{V_q}\Delta t \tag{4}$$

where H_{ref} denotes the reference pressure head, C_k is the coefficient of entrance loss, V_q is the fixed volume of the receiver and the subscript s stands for the step input (steady component of the inputs). Thus, K^+_{res} and K^+_{rt} are associated with the pressure head losses at the entrances of the sending reservoir and the receiver, respectively, and R^+ is related to the pipeline resistance coefficient. Generally, the initial pressure before the sudden pressure change is used as a reference pressure.

By solving for Q_P with the aid of the nondimensional parameters, the compatibility equations may be written in dimensionless form as

$$C^+;\ Q^+_P=C^+_A-H^+_P$$
$$C^-;\ Q^+_P=C^+_B+H^+_P \tag{5}$$
$$\text{where};\ C^+_A=(1-R^+_A)Q^+_A+H^+_A$$
$$C^+_B=(1-R^+_B)Q^+_B-H^+_B$$

The constants C^+_A and C^+_B are known for each time step. In Eq. (5), two unknowns, namely, H^+_P and Q^+_P, can be determined by solving these equations simultaneously : *i.e*., by first eliminating H^+_P in Eq. (5),

$$Q_P^+ = 0.5(C_A^+ + C_B^+) \tag{6}$$

then H_P^+ may be found directly from either one in Eq. (5). Thus, by using Eqs. (5) and (6), conditions at all interior points at a time step can be determined.

To illustrate the use of above equations, the pipeline is divided into n reaches, each having Δx $(= c\Delta t)$. The ends of these reaches are called sections. The end sections of the pipeline are referred to as boundaries.

3. Boundary Conditions

At either end of the transmission line, only one of the compatibility equations is available. Special boundary conditions are required to determine the transient head and discharge at the boundaries; *i.e.*, the tube entrance (sending reservoir) and the fixed volume termination (receiver). These are developed by solving the compatibility equations with the conditions imposed by the boundaries.

In the present analysis, it is assumed that the nondimensional pressure head level at the sending reservoir changes in a known manner after the sudden pressure change;

$$H_{res}^+ = H_s^+ + \Delta H^+ \sin \omega t \tag{7}$$

where ω is the circular frequency and $\Delta H^+ (= \Delta H/(H_s - H_{ref}))$ is the nondimensional amplitude of the sinusoidal disturbance. Let the entrance losses at tube entrance be given by

$$h_e = \frac{C_k Q_1^2}{2gA^2} \tag{8}$$

Then, the nondimensional pressure head at the entrance of the transmission line is expressed as

$$H_1^+ = H_{res}^+ - K_{res}^+ Q_1^{+2} \tag{9}$$

in which the subscript "1" stands for the entrance as the first node point in tube length. To develop the boundary condition, this equation is solved simultaneously with the negative characteristic equation in Eq. (5). By eliminating H_1^+ from the negative characteristic equation $(P=1)$ and the above equation, the following simplified equation is obtained

$$K_{res}^+ Q_1^{+2} + Q_1^+ - (C_2^+ + H_{res}^+) = 0 \tag{10}$$

for the flow rate at the entrance. Solving this equation and neglecting the negative sign with the radical term yield

$$Q_1^+ = \frac{-1 + \sqrt{1 + 4K_{res}^+ (C_2^+ + H_{res}^+)}}{2K_{res}^+} \tag{11}$$

by which the entrance pressure level, H_1^+, is determined from Eq. (9).

To develop the boundary condition at the fixed volume termination in a similar procedure shown for the entrance of the transmission line, the head loss at the exit end is written as

$$h_{rt} = \frac{C_k Q_n^2}{2gA^2} \tag{12}$$

and the pressure head is expressed in a dimensionless form as

$$H_n^+ = H_{rt}^+ - K_{rt}^+ Q_n^{+2} \tag{13}$$

in which n stands for the end of the transmission line, which means n reaches are used for calculation, and H_{rt}^+ denotes the nondimensional pressure head at the receiver. The discharge at the end of the pipeline is equal to the rate of increase within the receiver;

$$Au = \frac{V_q g}{c^2} \frac{dH_{rt}}{dt} \tag{14}$$

In the present analysis, it is assumed the changes of mass within the receiver occur adiabatically and reversibly. Thus, the nondimensional pressure head at receiver may be written as

$$H_{rt}^+ = H_{rt}^{+*} + D_a^+ Q_n^+ \tag{15}$$

In the above equation, the superscript * stands for the known value at the previous time step $(t^* = t - \Delta t)$. Eliminating H_{rt}^+ from the positive characteristic equation$(P=n)$ and from the above equation yields

$$K_{rt}^+ Q_n^{+2} - (1 + D_a^+) Q_n^+ + (C_{n-1}^+ - H_{rt}^{+*}) = 0 \tag{16}$$

By solving this equation and neglecting the positive sign with the radical term, the nondimensional discharge at the end of the pipeline is expressed as

$$Q_n^+ = \frac{(1 + D_a^+) - \sqrt{(1 + D_a^+)^2 - 4K_{rt}^+ (C_{n-1}^+ - H_{rt}^{+*})}}{2K_{rt}^+} \tag{17}$$

Now, the pressure heads at the receiver and the end of the pipeline, H^+_{rt} and H^+_n, may be determined from Eqs. (15) and (13), respectively. In the case of very small receiver sizes, the pipeline may be treated as a dead-ended transmission line. Since a particle speed at the boundary is zero, the positive characteristic equation at the dead end is written as $Q^+_n = C^+_{n-1} - H^+_n = 0$, from which the pressure head level at the end is determined; $H^+_n = C^+_{n-1}$.

4. Calculation Procedures and Numerical Results

Numerical results were calculated by choosing the working fluid (water or air), the geometry of the system (diameter and length of the pipe and end volume of the receiver), the input signal (unit step input and sinusoidal disturbance), the entrance head losses and the number of reaches. From the chosen values, the nondimensional parameters, K^+_{res}, K^+_{rt}, D^+_a, defined in Eq. (4), can be determined, and the local pipeline resistance coefficient, R^+, in a given time can be calculated for laminar or turbulent flows. In the calculations for R^+ and D^+_a, the time step is obtained by reach size and acoustic velocity, $\Delta t = \Delta x / c = L/nc$. In the present analysis,the entrance head loss coefficients are 0.5 for entrance flow from reservoir and 0.05 for exit flow to reservoir and acoustic flow velocities of 1500 m/s for water and 340 m/s for air are assumed, as a general example. The viscosities of water and air are assumed to be 9.13×10^{-5} $kg_f \cdot sec/m^2$ and 1.883×10^{-6} $kg_f \cdot sec/m^2$, respectively. The orders of the time step, used in the present analysis,are 10^{-3} for pneumatic system and 10^{-6} for hydraulic system; $n \approx 15$.

The fluid starts from a static condition ($H^+ = 0$, $Q^+ = 0$) and eventually ends in a harmonic disturbance conditions including static components. The nondimensional amplitude of the sinusoidal disturbance with respect to the static term is 0.2 in the present analysis. All interior conditions in the pipeline are determined by using Eqs. (5) and (6) with the known surrounding conditions calculated in the previous time step. The boundary conditions are calculated by using Eqs. (9) and (11) for sending end and Eqs. (13) and (17) for receiving end. The response to the input is determined by using Eq. (15). For the sake of brevity, only samples of the computational results are presented.

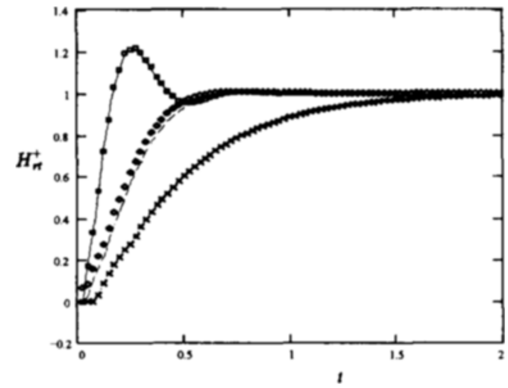


Fig. 2 Comparison of the existing analytical results (□, ◇, x) and the present numerical results (——, ----, -----) of the dynamic responses to the static step inputs for pneumatic systems, $D = 6.4$ mm:
—□—, $L = 10m$ $V_q = 2000cc$
---◇---, $L = 15m$ $V_q = 5000cc$
-----x-----, $L = 15m$ $V_q = 5000cc$

To verify the present numerical method, the method is applied to pneumatic transmission lines with static step inputs, where the analytical approach is applicable, and then the results are compared with the existing analytical results (Schuder and Binder, 1959). Present results showed good agreements with the existing results in Fig. 2. The effects of tube length, tube diameter and terminal volume on the response of pneumatic transmission lines to static inputs are shown in Fig. 3. The period of the peak-to-peak for discharge rate at the end of the transmission line can be estimated by considering the acoustic velocity and the tube length. The dynamic responses converges more or less within the period based on the lumped acoustic method, which will be shown later. As can be seen from the results, some general remarks are; (a) viscous damping increases with increasing tube length and decreasing tube diameter and (b) fluid-elastic stiffness generated by the compressibility of the fluid becomes predominant with decreasing terminal volume. The friction loss in pipeline will increase with increasing tube length and decreasing tube diameter. Fluid-elastic stiffness, due to the terminal volume, becomes larger with decreasing the volume such that the system is in underdamped motion because of the relatively high critical damping coefficients, defined by $c_c = 2\sqrt{mk}$. Here m and k denote mass and stiffness coefficient of the system. In other words, with increasing volume of the receiver, the damping becomes heavier as compared to other terms and the dynamic characteristics of the response can be similar to those of the overdamped system. It is found that the time of the first appearance of the pressure pulse at the terminal volume can be approximated by L/c.

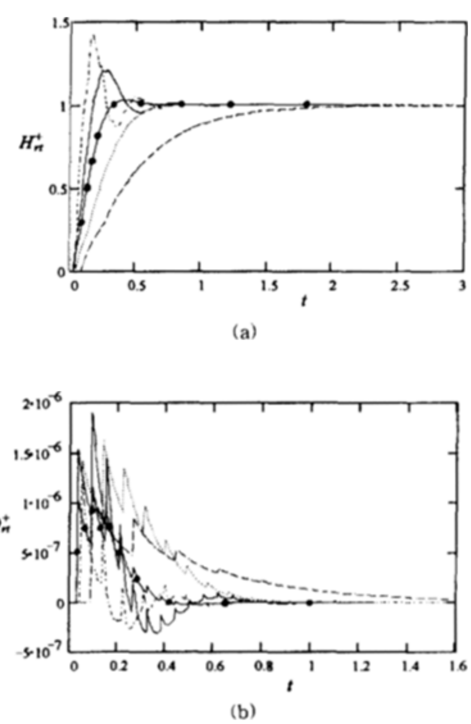


Fig. 3 (a) The dynamic responses to the static step inputs and (b) discharge rate at end of the transmission line for pneumatic systems:

——,	$L=10m$	$V_q=2000cc$	$D=6.4mm$
- - - -,	$L=30m$	$V_q=5000cc$	$D=6.4mm$
– • –,	$L=10m$	$V_q=2000cc$	$D=5.4mm$
·······,	$L=15m$	$V_q=5000cc$	$D=6.4mm$
– · –,	$L=15m$	$V_q=500cc$	$D=6.4mm$

In order to estimate the damping factor of the lumped acoustic element as known as a Helmholtz resonator with a relatively long wave length, $L_\lambda(=cT_l \gg L)$, the lumped acoustic impedance can be used. The mass and stiffness coefficients are written as $m=\rho L/A$ and $k=\rho c^2/V_q$, respectively. Thus, the natural frequency of the lumped acoustic system is $\omega_l = c\sqrt{A/(LV_q)}$. The damping force, F_d, acting on the element due to the friction head loss, ΔH_f can be approximated as

$$F_d = \rho g \Delta H_f A \tag{18}$$

where

$$\Delta H_f = f\frac{L}{D}\frac{u^2}{2g}$$

In the above equation, the time derivative of the mean velocity in Eq. (1) is neglected, since the natural frequency of the present pneumatic system is relatively small for long transmission line and large terminal volume. Thus, the damping factor of the lumped acoustic system, due to the friction head loss for the laminar flow can be expressed as

$$\zeta = \frac{16\mu}{\rho D^2 c}\sqrt{\frac{LV_q}{A}} \tag{19}$$

For the systems shown in Fig. 3, the length of the transmission line is shorter than the acoustic wave length. Thus it is possible to calculate the damping factor using the above equation. The approximated analytical results, which turns out to be reasonable, are presented in Table 1.

The axial variations of the pressure head and flow rate are depicted in Fig. 4 for a pneumatic system at various times during the half of the period based on acoustic velocity and tube length. It is clearly shown the acoustic wave moves from right to left and then from left to right. The

Table 1 The approximated damping factor due to the friction head loss.

$D(mm)$	$L(m)$	$V_a(cc)$	ζ
6.4	10	2000	0.432
	15	500	0.264
		5000	0.836
	30	5000	1.182
5.4	10	2000	0.718

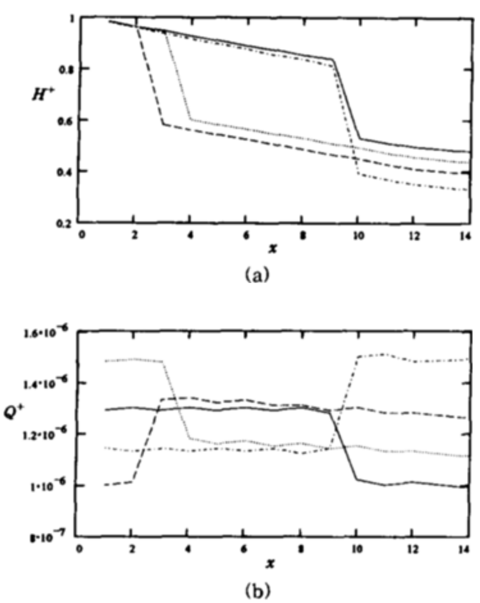


Fig. 4 Axial variations of (a) pressure head and (b) flow rate for the pneumatic system ($L=15m$, $V_q=5000cc$, $D=6.4mm$) at various times (- · -, $t=0.147$; ----, $t=0.165$; ·······, $t=0.182$ and ——, $t=0.2$ sec): Period based on lumped acoustic element method, $T_l=0.691$ sec; period based on acoustic velocity, $T_a=0.171$ sec.

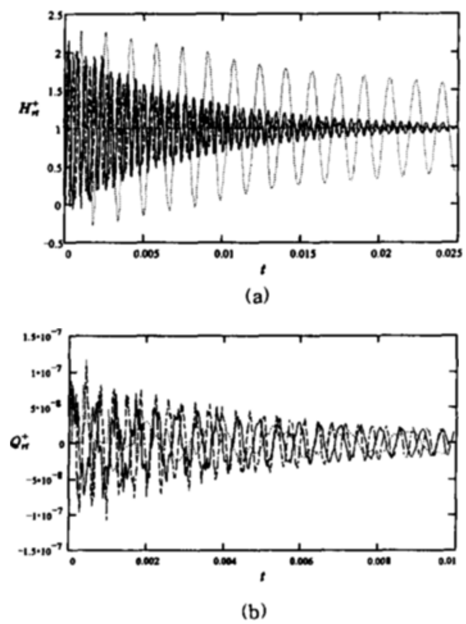


Fig. 5 (a) Typical responses to the static step inputs and (b) discharge rate at the end of the transmission line for hydraulic systems ($D=3.2mm$, $V_q=1$ cc):
——, $L=0.05m$; - - - -, $L=0.1m$;
······, $L=0.5m$

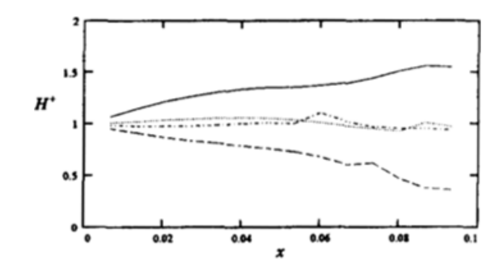
Table 2 The amplitude ratio of peak pressure to static step input and its response time in bracket, $D=6.4$ mm.

L	$V_q(cc)$				
(m)	2000	800	500	200	80
10	1.210 (0.275)	1.418 (0.165)	1.433 (0.154)	1.762 (0.099)	1.844 (0.091)
20	1.075 (0.440)	1.237 (0.235)	1.373 (0.204)	1.545 (0.185)	1.598 (0.181)
30	1.014 (0.694)	1.137 (0.325)	1.245 (0.296)	1.379 (0.275)	1.427 (0.269)

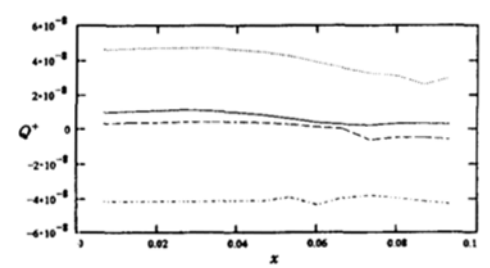
pressure head at right end near the terminal volume is continuously increased with time, while the discharge rate is decreased. The preassure is decreased along the axial direction, mainly due to the viscous skin friction. This is the main reason why the damping factor can be estimated by Eq. (19). If the pressure loss is strongly influenced by the time derivation of mean velocity, its use is questionable-see Eq. (1).

The amplitudes of peak pressure of response to the static step inputs and the corresponding peak-pressure response times from $t=0$ in bracket are shown in Table 2. The times of the first appearance of the pressure pulse are 0.029 sec for $L=10$ m, 0.059 sec for $L=20$ m, and 0.088 sec for $L=30$ m. For the relatively large terminal volume, the response converges to the static step input without peak pressure, which means the system is overdamped. In general, the fluid-elastic stiffness term due to the compressibility of the fluid becomes larger with decreasing the terminal volume, and inertia terms of the mass in pipeline becomes larger with the tube length. Thus, it is physically true that the peak-pressure response time, related to the natural frequency of the system, becomes shorter with decreasing tube length and terminal volume.

Typical responses of the hydraulic transmission lines to the static step inputs and the discharge rate at the terminal volume are shown in Fig. 5. In the case of water, the fluid-elastic stiffness of the terminal volume is predominant to the damping of the instrument system because of a relatively high bulk modulus of elasticity of water, as compared to that of air. In this case, it is required to have a very large volume receiver to see the dynamic characteristics of the overdamped system, which is not realistic for a hydraulic instrument system. Moreover, the inertia term for water is relatively larger compared to that of air. As a result, the critical damping coefficient of the hydraulic system easily overcomes the viscous damping coefficient. This is the main reason why the response of the instrument system, shown in the present analysis for water, has the underdamped characteristics. As shown in the results for pneumatic transmission lines, the response converges more slowly to the static end condition with increasing the tube length.



(a)



(b)

Fig. 6 Axial variations of (a) pressure head and (b) flow rate for the hydraulic system ($L=0.1\,m$, $V_q=1\,cc$, $D=3.2\,mm$) at various times (– · –, $t=0.00463$; ----, $t=0.00475$; ·····, $t=0.00488$ and ——, $t=0.005$ *sec*): Period based on lumped acoustic element method, $T_l=0.000512$ *sec*; period based on acoustic velocity, $T_a=0.000267$ *sec*.

In Fig. 6, the axial variations of the pressure head and the flow rate are shown for a hydraulic system at various times during the period based on the lumped acoustic element method. As compared to the result for air, the change of pressure head or flow rate through the acoustic wave is very small and the pressure loss along the axial direction is influenced by the time derivative of the mean flow velocity. It is found that the phase difference between flow rate and pressure head is about 90°, which means the pressure head is in maximum value when the flow rate is about zero. The frequency of the dynamic response is strongly influenced by the lumped acoustic element system. It is found that the effect of the entrance or discharge head loss is minor even though the length of the pipe is relatively short.

As an indication of a degree of damping for hydraulic system, the damping ratio of the pressure head at the end of the transmission line was primarily used. This gives a measure of the peak-to-peak attenuation of the pressure oscillation. For the present hydraulic system with small terminal volume and short pipe line, the damping factor can not be calculated by Eq. (19), because the effect of the time derivative of the mean velocity is significant in the momentum equations for hydraulic system. However, the natural frequency can be estimated by the lumped acoustic element method. The mean damping factor, presented in this paper for the hydraulic system, is defined as

$$\zeta_{mean}=\frac{1}{2\pi J}\sum_{j=1}^{J}\ln\left(\frac{H_{rt}^{+(j)}-1}{H_{rt}^{+(j+1)}-1}\right)\quad \text{for } \zeta_{mean}\ll 1 \tag{20}$$

Since the system is not exactly linear, the mean values of the damping factor and the frequency are calculated considering the first 30 peak-to-peak pressures. As shown in Table 3, the approximated mean damping factor is increased and the mean natural frequency is decreased with length of the line. For a relatively long pipe with small receiver volume, however, the damping factor becomes smaller, which might be due to the effect of the dead end. It is interesting to note that the damping factor of the pressure for the present hydraulic systems becomes smaller with increasing terminal volume and decreasing tube diameter. This trend reverses the effects of tube diameter and terminal volume given for air. The mean damping factor is quite larger than that calculated by Eq. (19), which is not shown here.

Table 3 The mean damping factor and natural frequency and the approximated natural frequency by the lumped element method, for hydraulic transmission line.

$D(mm)$	$V_a(cc)$	$L(m)$	ζ_{mean}	ω_{mean}	ω_l
3.2	1	0.01	0.0046	41,930	42,538
		0.05	0.0089	17,870	19,023
		0.1	0.0094	12,270	13,451
		0.5	0.0086	3,800	6,015
	5	0.01	0.0021	18,960	19,023
		0.05	0.0047	8,390	8,507
		0.1	0.0064	5,850	6,015
		0.5	0.0098	2,450	2,690
2.0	1	0.01	0.0031	26,440	26,580
		0.05	0.0062	11,990	11,889
		0.1	0.0085	8,260	8,407
		0.5	0.0129	3,010	3,760

In Fig. 7, the responses, obtained by the present and existing methods for relatively long tube length and small terminal volume (although, this kind of system might be not used in a practical instrument system), are presented and the results are compared to the calculated result by the present method for dead end. The peak pressure of the response is approximately twice of the static input, which means that the wave is reflected with only a slight reduction in amplitude and no change in phase. This phenomenon is physically true for the dead end of the acoustic device. Good agreement was found between the results.

To investigate the dynamic response to a sudden pressure change containing sinusoidal disturbance, the present method is applied to the pneumatic instrument systems of relatively long pipeline and the hydraulic instrument systems of rela-

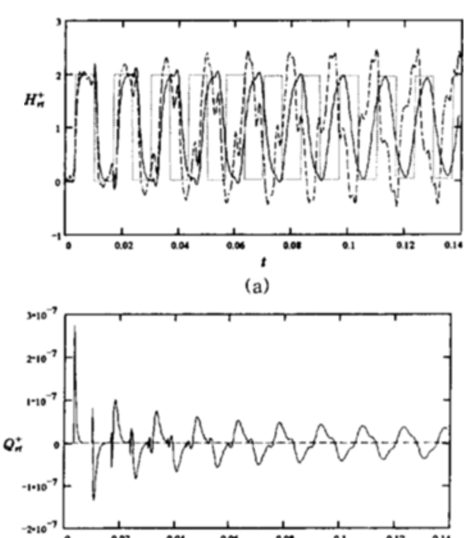


Fig. 7 Comparison of (a) the responses to the static step inputs (b) discharge rate at end of the transmission line for relatively long hydraulic system ($L=5m$, $D=6.4mm$), obtained by the present numerical method for $V_q=20cc$ (——), $V_q=0$ cc (·····) and the existing analytical method for $V_q=20cc$ (- - - -).

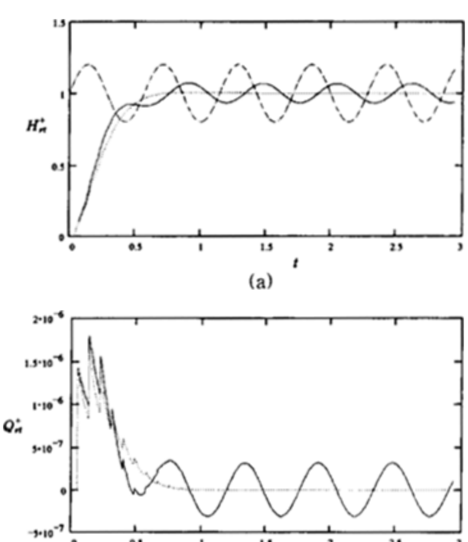


Fig. 8 (a) Typical dynamic response (——) to the step input including sinusoidal disturbance (- - - -, $H_s^+=1$ $\Delta H^+=0.2$ $\omega=11 rad/\text{sec}$) and a typical response (·····) to the static step input and (b) discharge rates at end of the transmission line for pneumatic system ($L=15m$ $D=6.4mm$ $V_q=5000cc$)

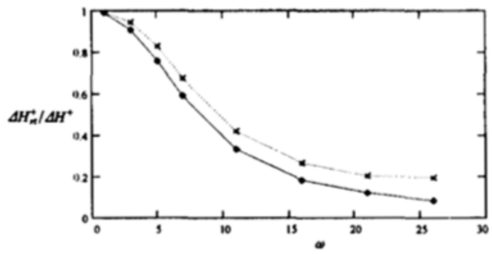


Fig. 9 Attenuation-frequency curves for pneumatic systems, $D=6.4mm$:
—◇—, $L=30m$ $V_q=2000cc$
···· x····, $L=15m$ $V_q=5000cc$

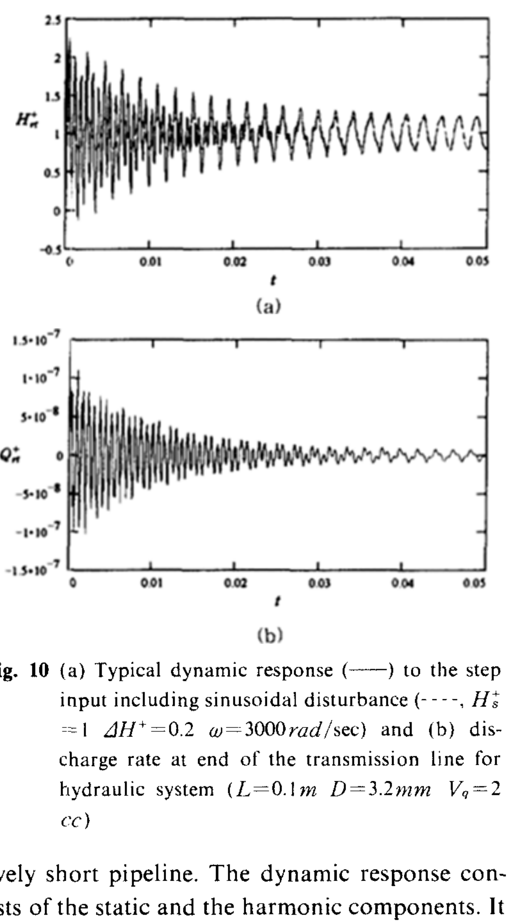


Fig. 10 (a) Typical dynamic response (——) to the step input including sinusoidal disturbance (- - - -, $H_s^+=1$ $\Delta H^+=0.2$ $\omega=3000 rad/\text{sec}$) and (b) discharge rate at end of the transmission line for hydraulic system ($L=0.1m$ $D=3.2mm$ $V_q=2cc$)

tively short pipeline. The dynamic response consists of the static and the harmonic components. It is of interest to evaluate the amplitude ratio of the harmonic response to the sinusoidal input, especially for the long pneumatic instrument system. The typical result is shown in Fig. 8 for a pneumatic instrument system. As shown in the previous static-response, the response converges to the dynamic end conditions. The characteristics of the static component is similar to that of the static response shown before. It was found that the amplitude ratio and phase lag of the harmonic response to the sinusoidal input were 0.325 and 2.156 rad, respectively. Thus, it is desired to obtain the amplitude ratio versus frequency as shown in Fig. 9. These dynamic responses, as a result of frequency response analysis, could not be used directly to predict the other dynamic responses, since the system was nonlinear. However, rough approximations over a limited range could be obtained by the methods used in linear analysis. The results indicate that the harmonic responses to the sinusoidal disturbances are attenuated with increasing frequency of sinusoidal disturbances.

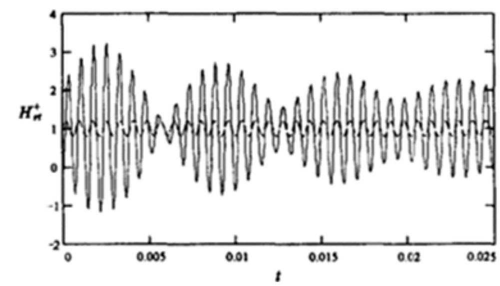


Fig. 11 Dynamic resonance of the harmonic response (——) to the step input including sinusoidal disturbance (- - - -, $H_s^+=1$ $\Delta H^+=0.2$ $\omega=8000$ rad/sec) for hydraulic system ($L=0.1m$ $D=3.2$ mm $V_q=2cc$), as a Helmholtz resonator.

As can be seen in Fig. 10 for the hydraulic instrument system of relatively short pipeline, the amplitude of the harmonic response is almost equal to the sinusoidal inputs. The phase lag to the input does not exist because of the relatively short pipeline. If the wave length in the fluid is much longer than the dimensions of the device, analysis of acoustic devices becomes simple; *e.g.*, for hydraulic instrument system with relatively short pipeline. As shown in Fig. 11, resonance occurs when the excitation frequency of the sinusoidal disturbance is approximately equal to the natural frequency, $\omega_l=c\sqrt{A/(LV_q)}$, of the lumped acoustic system. The amplitude ratio of the harmonic response is found to be approximately triple of that of the sinusoidal inputs.

5. Conclusions

A numerical approach, for evaluating the response of transmission lines to a sudden pressure changes including sinusoidal inputs, has been developed. This analysis is based on method of characteristics, which is used to transform the partial differential equations into the ordinary differential equations integrable along the characteristic lines. The dynamic characteristics of dead-ended or volume-terminated transmission lines have been studied. Numerical results were calculated for pneumatic and hydraulic systems. The governing equations including boundary conditions for compressible fluid flow have been developed. The results obtained by application of the equations should provide adequate information for design of the system. The present numerical approach has less restrictions and less empirical correction factors, which arise from an analytical approach. To verify the present numerical approach, the present results have been compared with those of the existing analytical approach, where the both approaches can be applicable. The agreement between the results was found to be satisfactory.

To indicate the degree of damping on the pressure head at the end of the transmission line, an analytical method is developed based on the viscous pressure loss due to the skin friction. It is found the method might be useful for pneumatic system with long tube. For hydraulic system, logarithmic decrement method is used to evaluate the damping factor. For the present pneumatic system, the viscous damping through the transmission pipeline becomes larger with increasing pipeline length and decreasing diameter of the pipe and the fluid-elastic stiffness decreases with increasing the terminal volume. However, the effect of tube diameter and terminal volume on the response reverses this trend for the hydraulic system. The viscous damping coefficient for relatively long pipe can overcome the critical damping coefficient with increasing the terminal volume, due to the relatively small fluid-elastic stiffness for pneumatic system.

By inspection of the present results, some general conclusions are found; (1) it is required for pneumatic instrument systems to use relatively long transmission pipeline and large terminal volume in order to have an overdamped dynamic characteristic of the response to the step input, (2) amplitude ratio of the harmonic response to harmonic input decreases with frequency of the input for pneumatic system, (3) the response of pressure to the static step input is converged more or less within a period given by the lumped acoustic element method, (4) a relatively long transmission pipeline is not suitable for hydraulic instrument system because of waterhammer in the pipeline, (5) the frequency can be estimated by the lumped acoustic element method for hydraulic system and (6) resonance on the harmonic response due to the lumped acoustic impedance occurs for hydraulic system. It is noted the effect of the entrance and discharge head loss is minor even though the transmission line is relatively short.

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Appendix

The response of pneumatic transmission lines to the step input has been conducted analytically, based on a one dimensional uniformly distributed system, small, reversible, adiabatic-pressure changes and laminar flow, by Schuder and Binder (1959). The two basic differential equations governing the transient flow in pipes are

$$\frac{\partial u}{\partial x} = -\frac{1}{\rho c^2}\frac{\partial p}{\partial t}$$

$$\frac{\partial u}{\partial x} = -\rho\frac{\partial u}{\partial t} - f_R u \qquad \text{(a)}$$

subjected to boundary conditions at the sending reservoir and the receiver;

$$\frac{p-p_0}{P_m-p_0} = 1$$

$$\rho A u = V_q\frac{d\rho}{dt} \qquad \text{(b)}$$

where $f_R\left(=\frac{32\mu}{D^2}\right)$ is the tubing resistance.

Using Laplace transformation of the governing equations including the boundary conditions,the solution of the system was obtained. With inverse transformation, a complete picture of pressure as a function of x and t is expressed as

$$\frac{p-p_0}{P_m-p_0} = 1-2e^{\frac{-f_R T}{2\rho}}\sum_{\alpha_1}^{\alpha_\infty}\frac{\cos\frac{\theta t}{2}+\frac{f_R}{\rho\theta}\sin\frac{\theta t}{2}}{\alpha\left[\left(\frac{V_q}{AL}+1\right)\sin\alpha+\frac{V_q\alpha}{AL}\text{cas }\alpha\right]}$$

$$\text{for } \frac{f_R}{L}<\frac{2\alpha c}{L} \qquad \text{(c)}$$

$$\frac{p-p_0}{P_m-p_0} = 1-2e^{\frac{-f_R T}{2\rho}}\sum_{\alpha_1}^{\alpha_\infty}\frac{\cosh\frac{\theta t}{2\iota}+\frac{f_R l}{\rho\theta}\sinh\frac{\theta t}{2\iota}}{\alpha\left[\left(\frac{V_q}{AL}+1\right)\sin\alpha+\frac{V_q\alpha}{AL}\text{cas }\alpha\right]}$$

$$\text{for } \frac{f_R}{L}>\frac{2\alpha c}{L} \qquad \text{(d)}$$

where $\theta=\left[\left(\frac{2\alpha c}{L}\right)^2-\left(\frac{f_R}{L}\right)\right]^{1/2}$, $\alpha\tan\alpha=\frac{AL}{V_q}$

The value of α may be obtained by a numerical method; *e.g.*, the secant method.